Markov Chain Monte Carlo, Numerical Integration
(See Statistics)

Trevor Gallen

Fall 2015

Agenda

- Numerical Integration: MCMC methods
- Estimating Markov Chains
- Estimating latent variables

Numerical Integration: Part II

- Quadrature for one to a few dimensions feasible for well-behaved distributions
- For many-dimensional integrals, we typically use Markov chain Monte Carlo
- There are many different methods
- I discuss a simple one (Gibbs Sampling) and a more complex one (Metropolis-Hastings)
- This is a prominent problem in Bayesian analysis

The Problem

- Say your data is summarized by a two-dimensional problem: height and weight $f(x_1, x_2)$
- You want a population, for $i \in (1, ..., n)$, (x_1^i, x_2^i)
- You have access to the *conditional* marginal distributions. That is, while $f(x_1, x_2)$ is ugly, $f(x_1|x_2)$ and $f(x_2|x_1)$ are easy to sample from.

Gibbs Sampling

1. Start with (x_1^0, x_2^0)
2. Sample $x_1^1 \sim f(x_1|x_2^0)$
3. Sample $x_2^1 \sim f(x_2|x_1^1)$
4. We have a LLN and CLT that states that:

$$\frac{1}{N}\sum_{i=1}^{N} g(x^i) \to \int g(x)f(x)dx$$

Gibbs Sampling: Example

- Assume a distribution:

$$x \sim \mathcal{N}(0, \Sigma) \quad \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

- Then we can get the conditional marginal distributions:

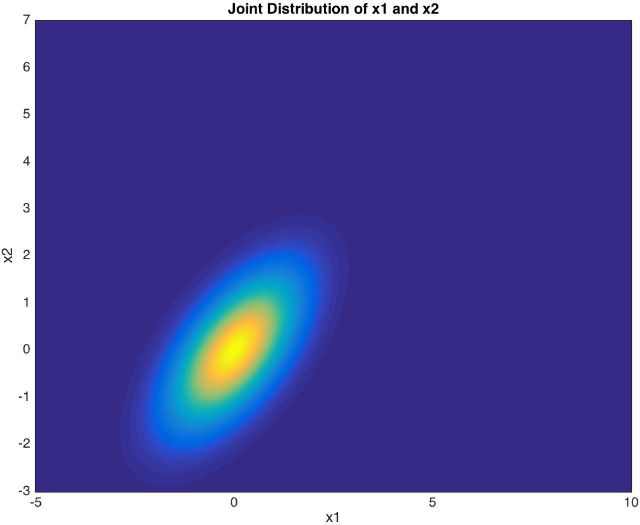
$$x_1|x_2 \sim \mathcal{N}\left(\rho x_2, (1-\rho)^2\right)$$

$$x_2|x_1 \sim \mathcal{N}\left(\rho x_1, (1-\rho)^2\right)$$

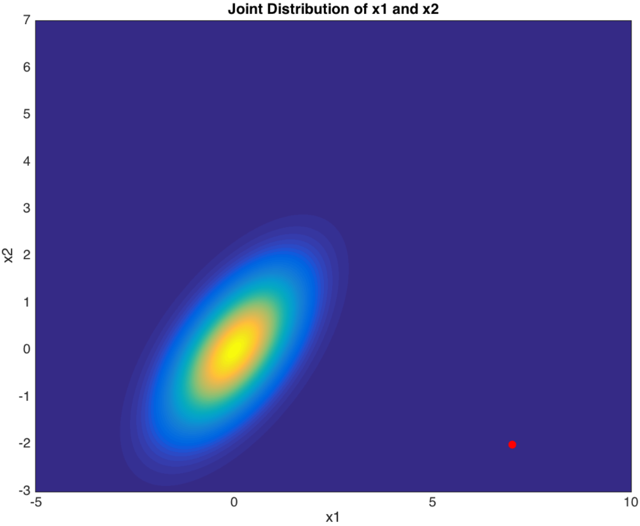
- Iteratively sample from these.

- See Gibbs.m

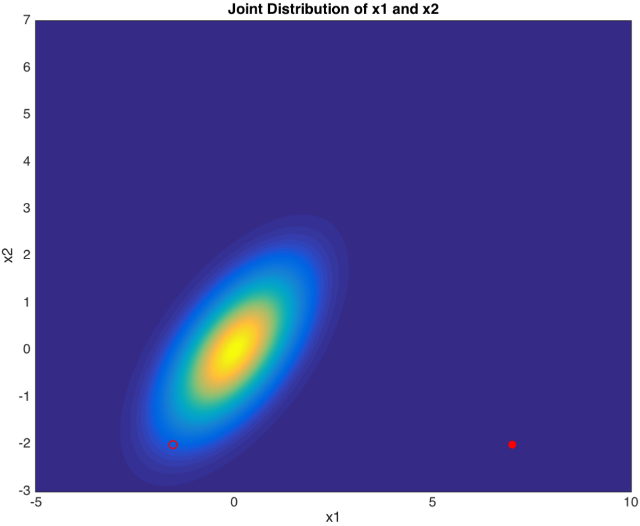
Gibbs Sampling: Example



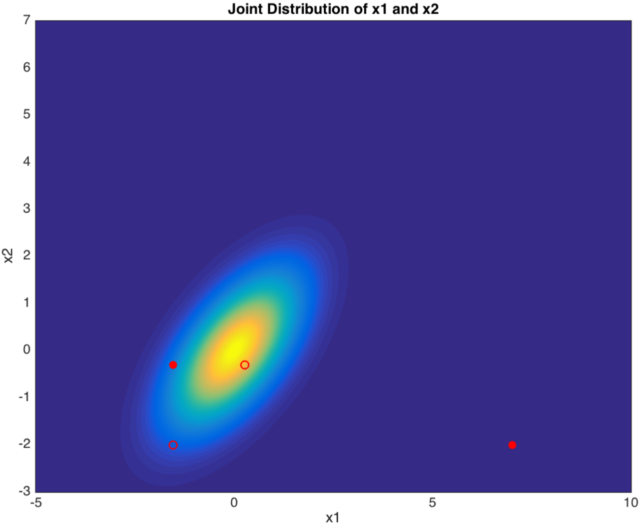
Gibbs Sampling: Example



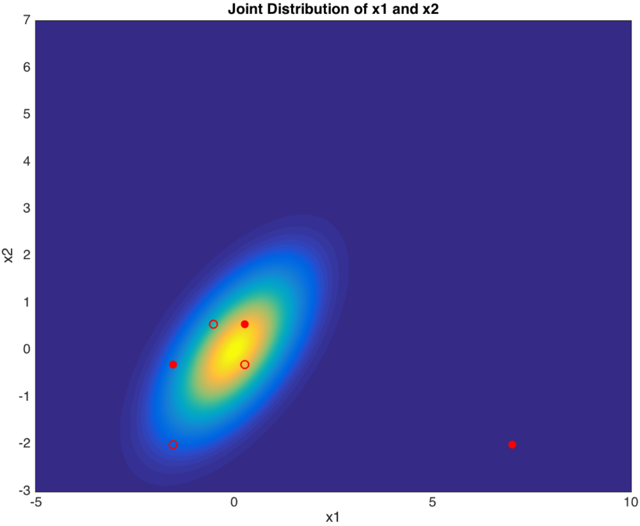
Gibbs Sampling: Example



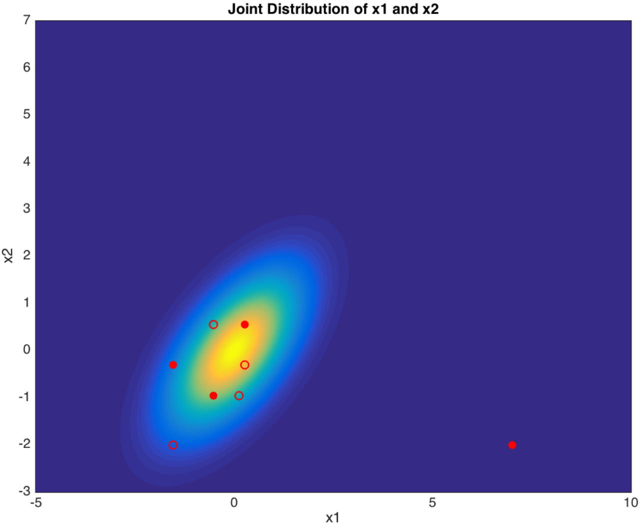
Gibbs Sampling: Example



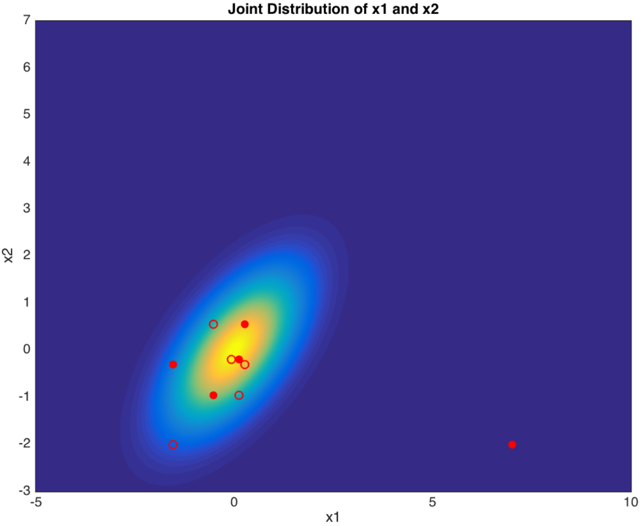
Gibbs Sampling: Example



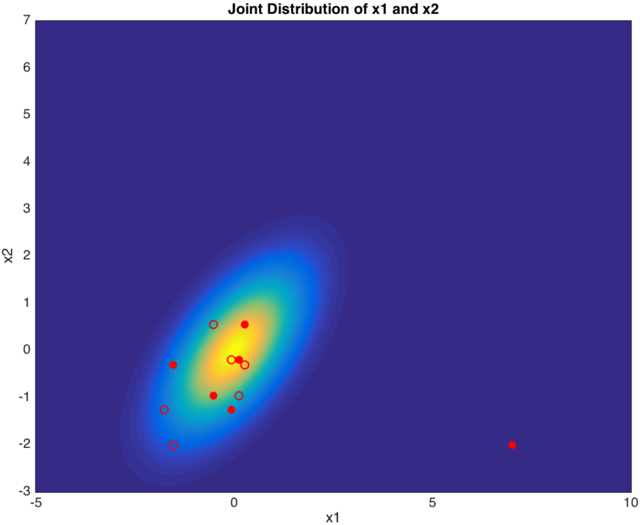
Gibbs Sampling: Example



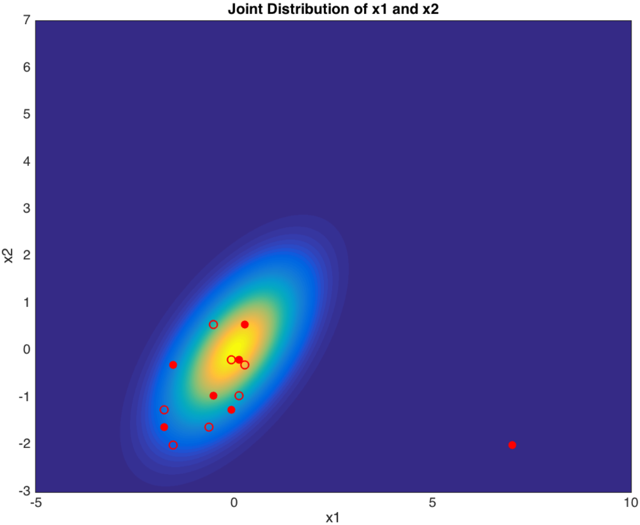
Gibbs Sampling: Example



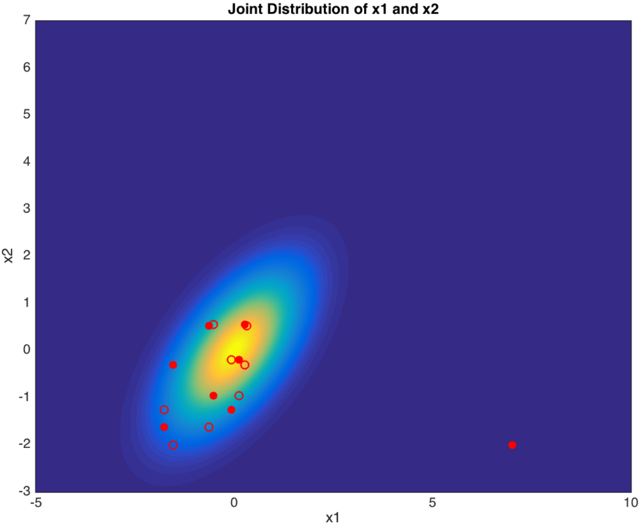
Gibbs Sampling: Example



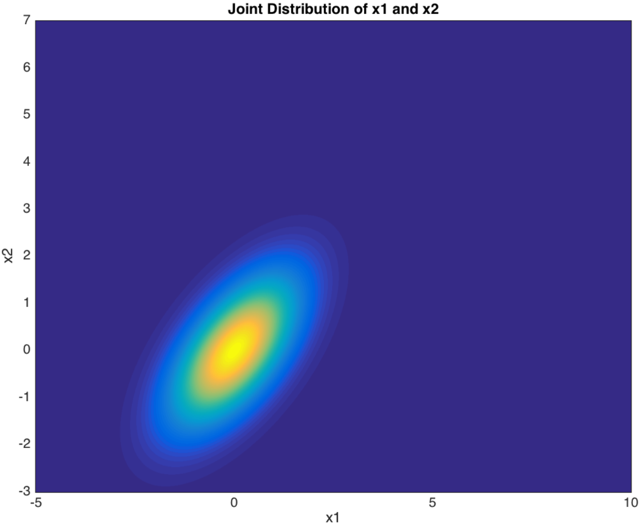
Gibbs Sampling: Example



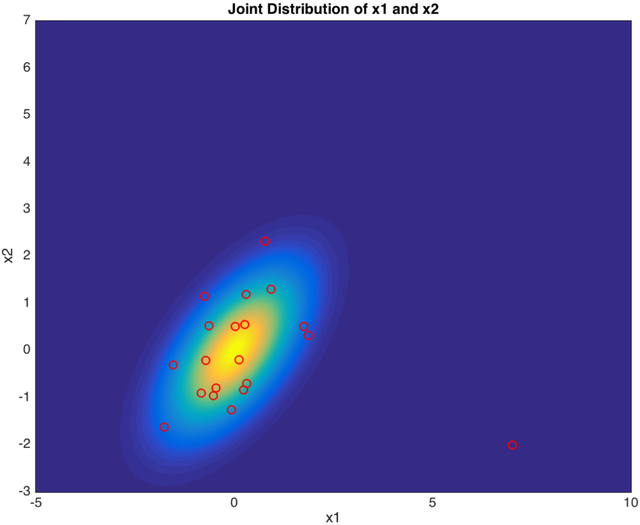
Gibbs Sampling: Example



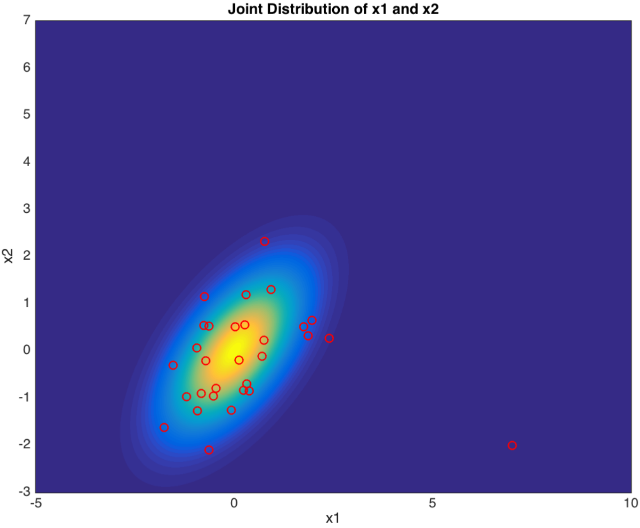
Gibbs Sampling: Example



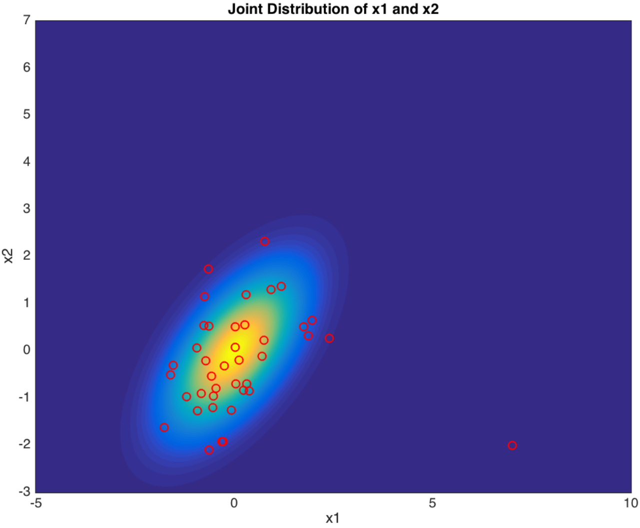
Gibbs Sampling: Example



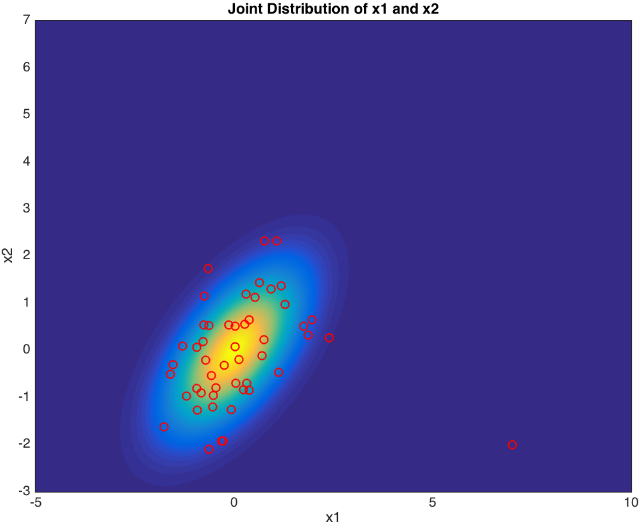
Gibbs Sampling: Example



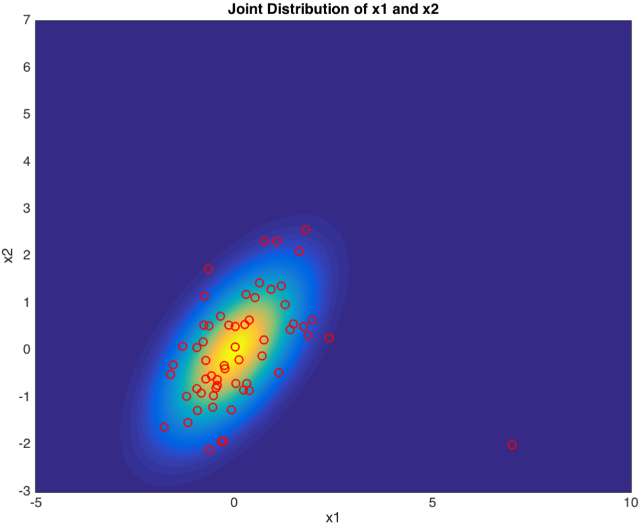
Gibbs Sampling: Example



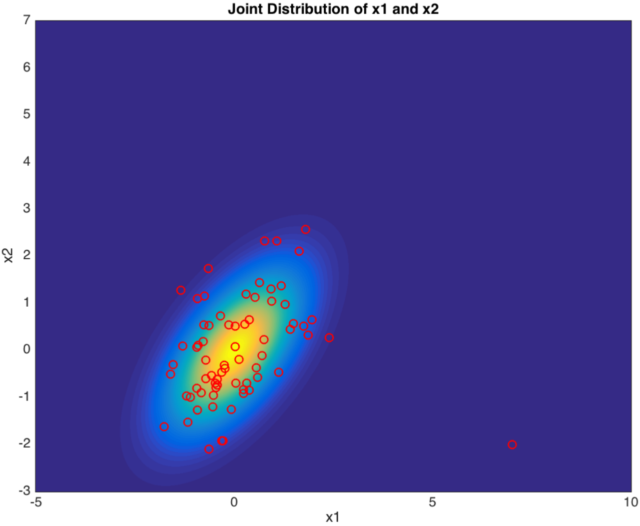
Gibbs Sampling: Example



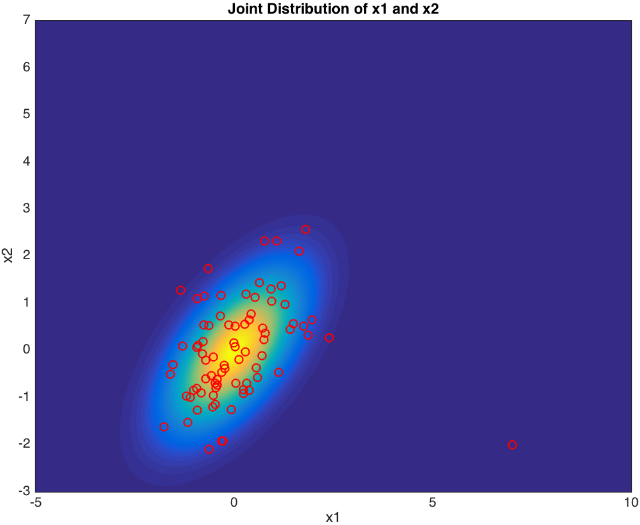
Gibbs Sampling: Example



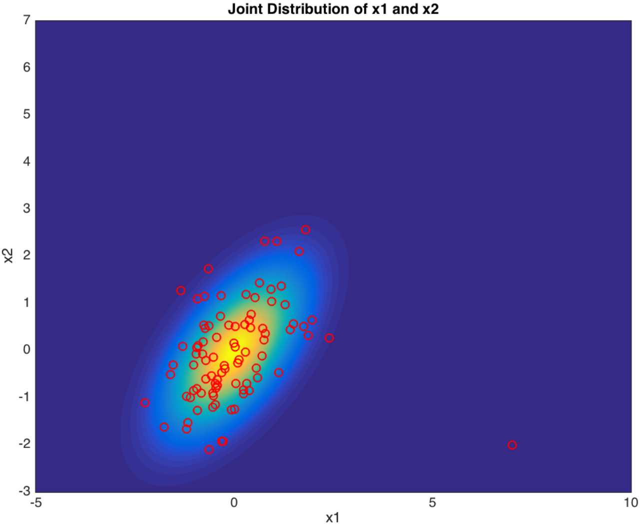
Gibbs Sampling: Example



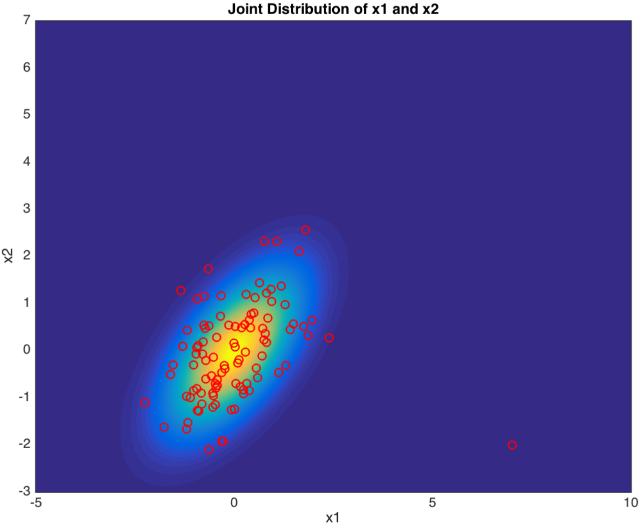
Gibbs Sampling: Example



Gibbs Sampling: Example



Gibbs Sampling: Example



Metropolis Hastings

What if we can only evaluate likelihood at a given point?

1. We start with some (multi-dimensional) value x^i and a proposal distribution $g(x|x^i)$

2. Grab a new sample from our proposal distribution:

$$x' \sim g(x|x^i)$$

3. Calculate acceptance probability:

$$pr(x^i, x') = \min\left\{1, \frac{f(x')}{f(x^i)} \frac{g(x^i|x')}{g(x', x^i)}\right\}$$

4. Accept the new value with probability $pr(x^i, x')$, otherwise, stay there.

5. This again converges in distribution to the true distribution.

A convenient proposal density

- If our proposal density is symmetric

$$g(x^i|x') = g(x', x^i)$$

- This is called random-walk Metropolis-Hastings

- Our acceptance probability is easy:

$$pr(x^i, x') = \min\left\{1, \frac{f(x')}{f(x^i)}\right\}$$

Metropolis-Hastings: Example

- Let's say our distribution is one-dimensional:

$$x \sim 0.5U(0,1) + 0.25U(-1,2) + 0.25U(0.5,0.75)$$

- Choose sampling distribution centered around current point:

$$g(x'|x) \sim \mathcal{N}(x, 0.1)$$

Sampling PDF

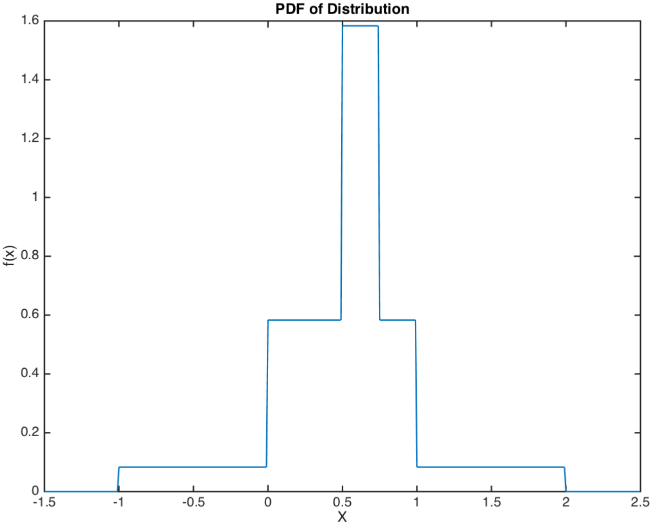
- Let's say our distribution is one-dimensional:

$$x \sim 0.5U(0,1) + 0.25U(-1,2) + 0.25U(0.5,0.75)$$

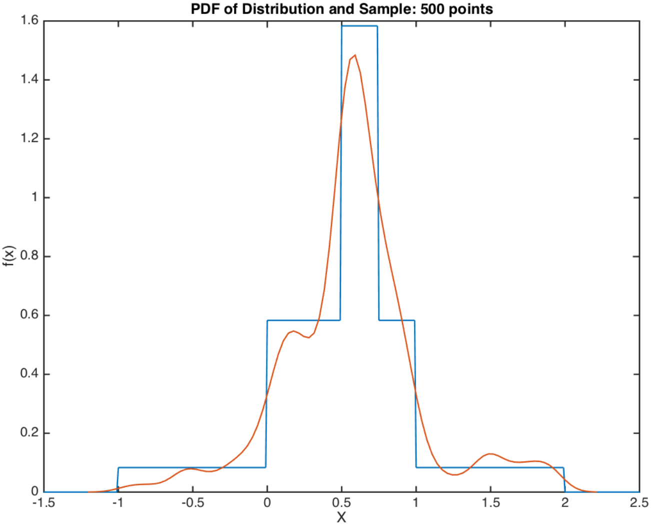
- Choose sampling distribution centered around current point:

$$g(x'|x) \sim \mathcal{N}(x, 0.1)$$

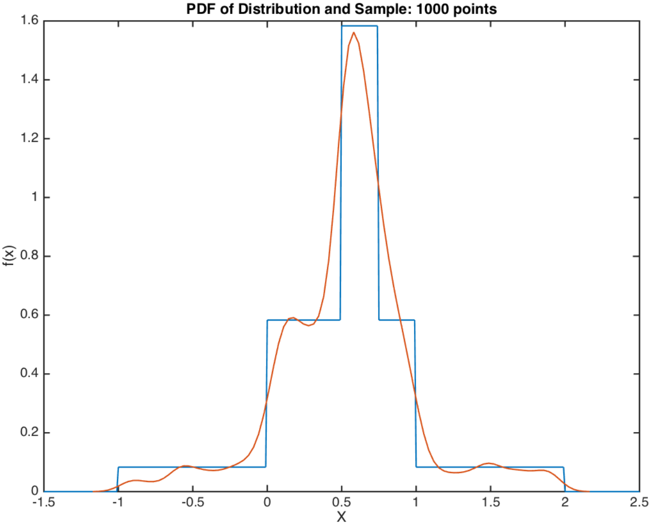
Metropolist-Hastings Random Walk: Example



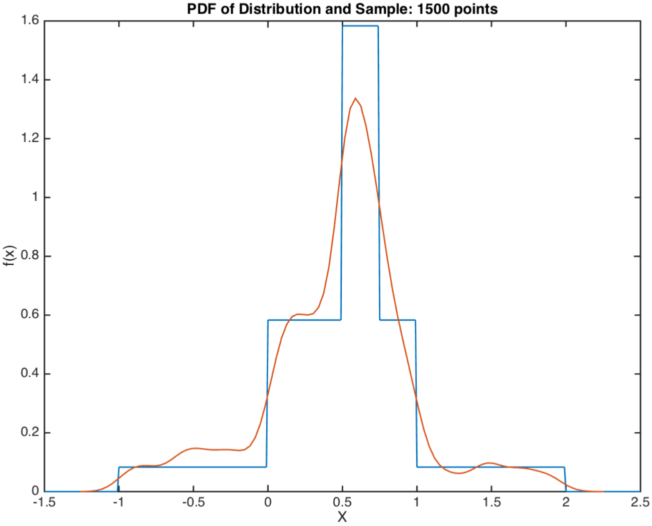
Metropolist-Hastings Random Walk: Example



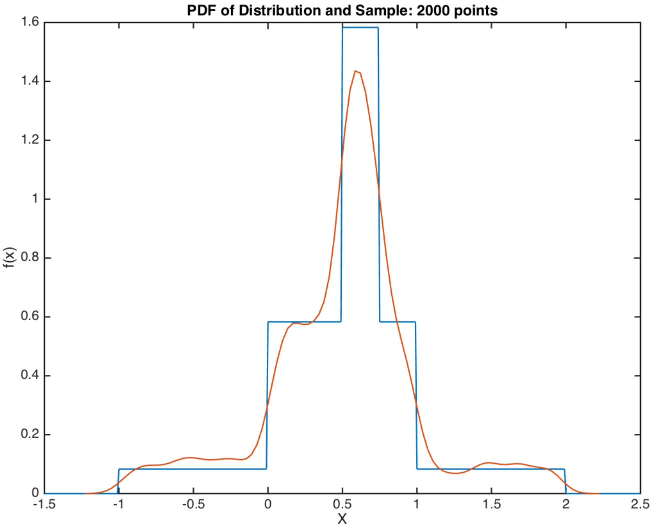
Metropolist-Hastings Random Walk: Example



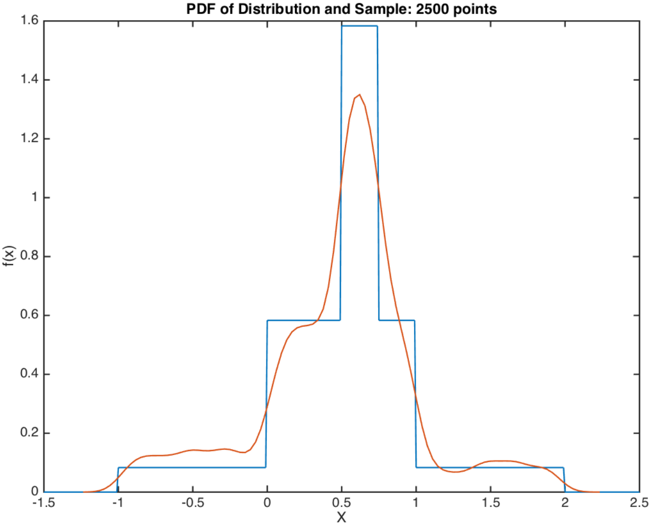
Metropolist-Hastings Random Walk: Example



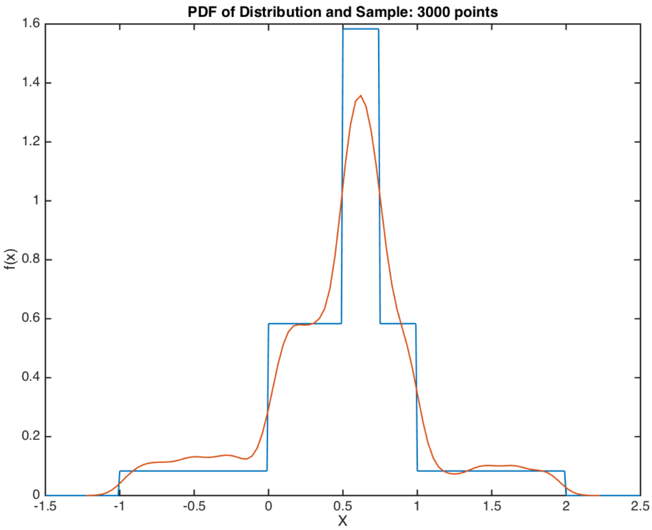
Metropolist-Hastings Random Walk: Example



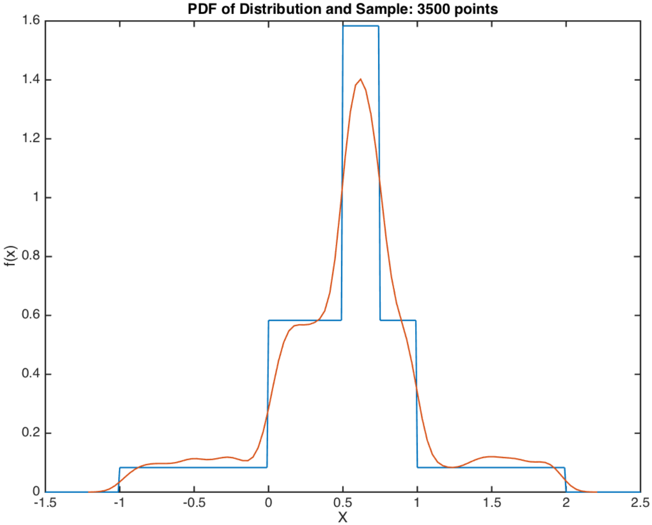
Metropolist-Hastings Random Walk: Example



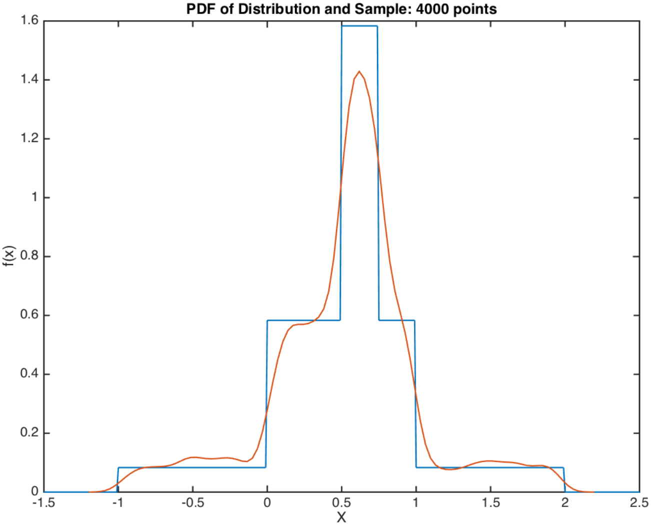
Metropolist-Hastings Random Walk: Example



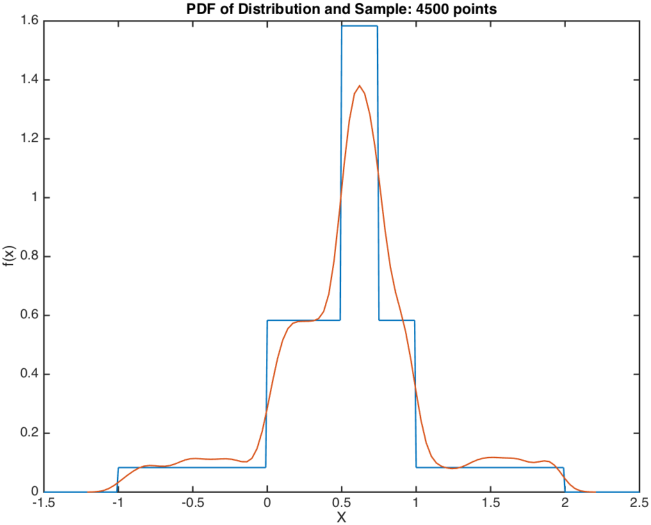
Metropolist-Hastings Random Walk: Example



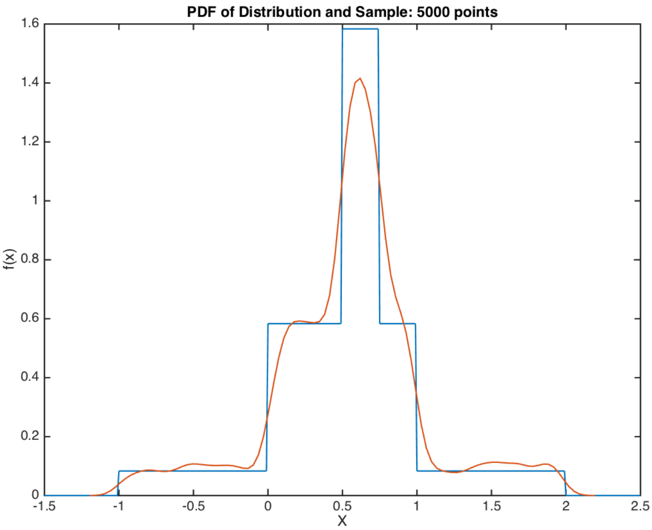
Metropolist-Hastings Random Walk: Example



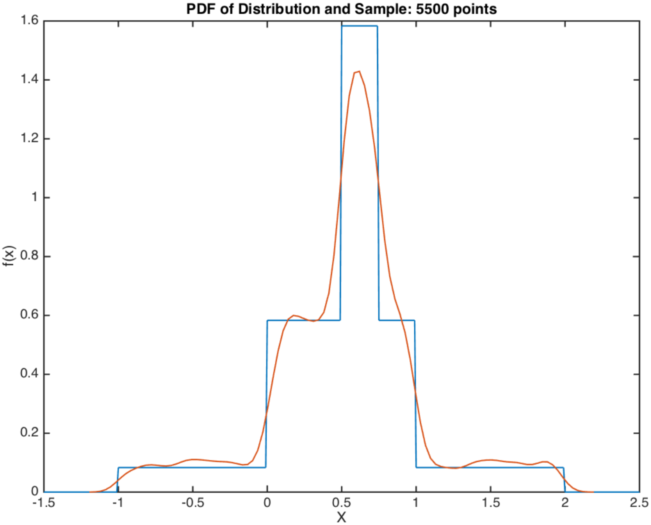
Metropolist-Hastings Random Walk: Example



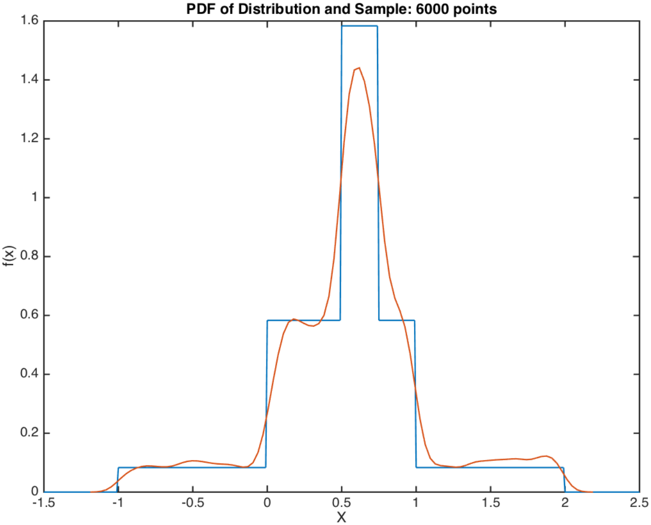
Metropolist-Hastings Random Walk: Example



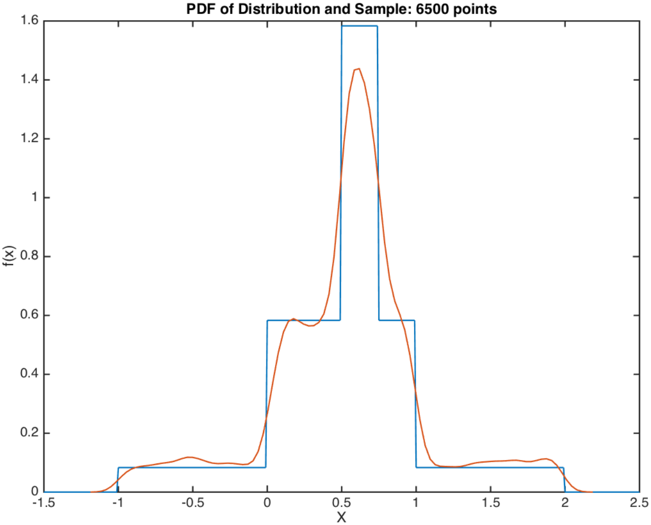
Metropolist-Hastings Random Walk: Example



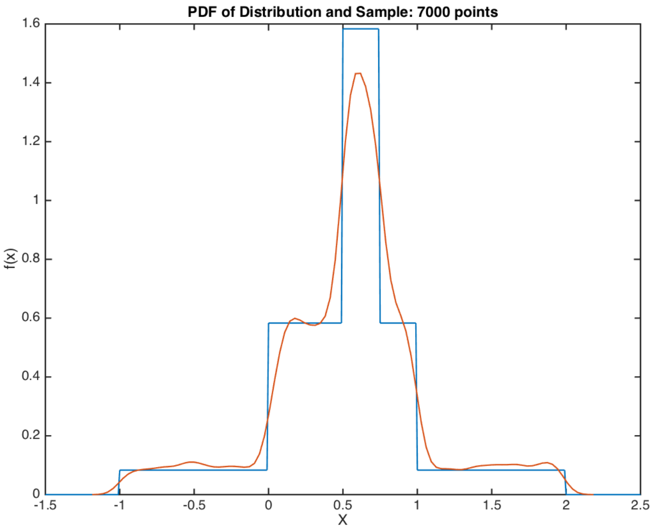
Metropolist-Hastings Random Walk: Example



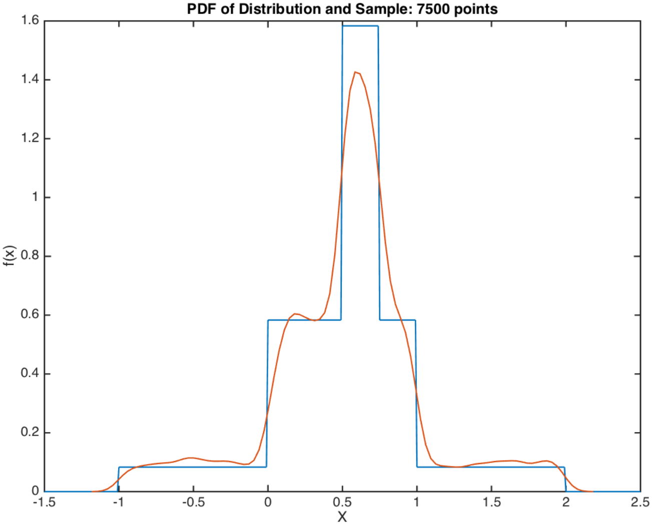
Metropolist-Hastings Random Walk: Example



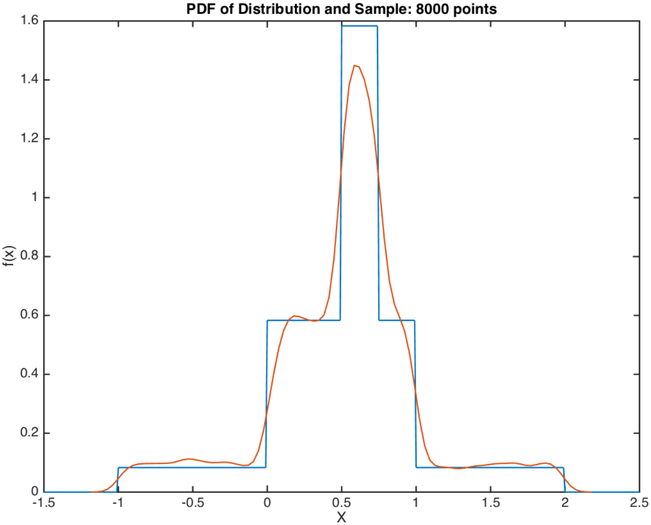
Metropolist-Hastings Random Walk: Example



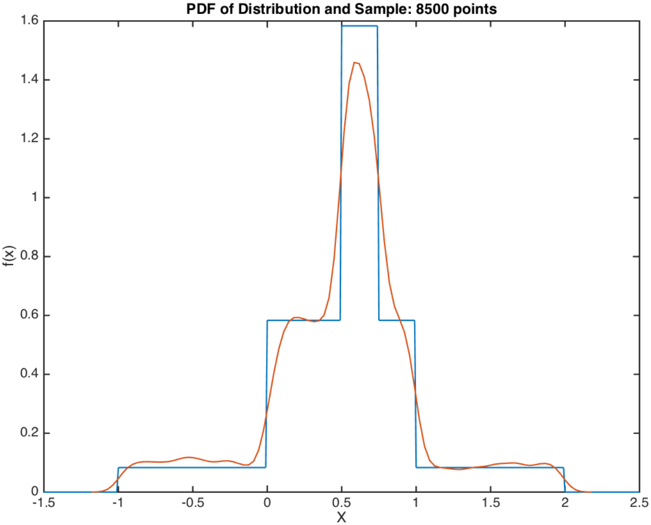
Metropolist-Hastings Random Walk: Example



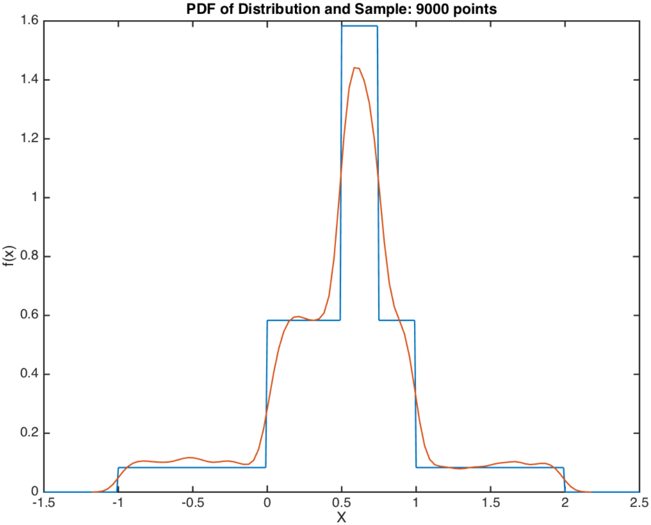
Metropolist-Hastings Random Walk: Example



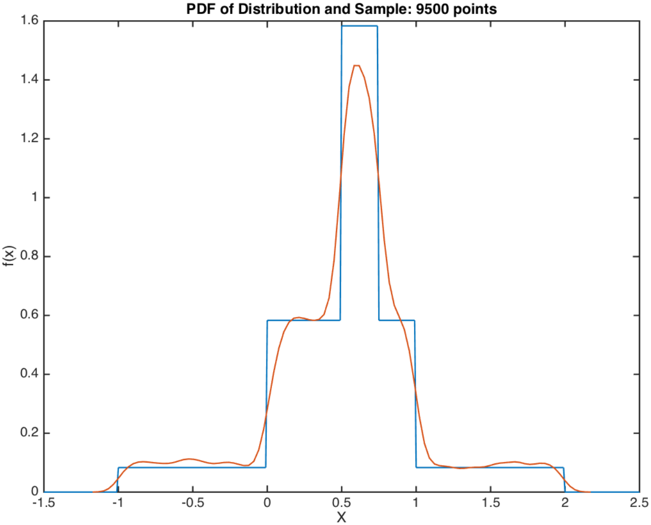
Metropolist-Hastings Random Walk: Example



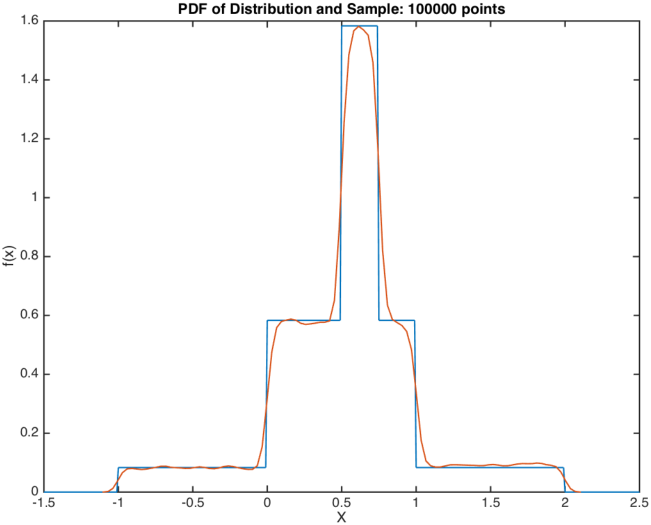
Metropolist-Hastings Random Walk: Example



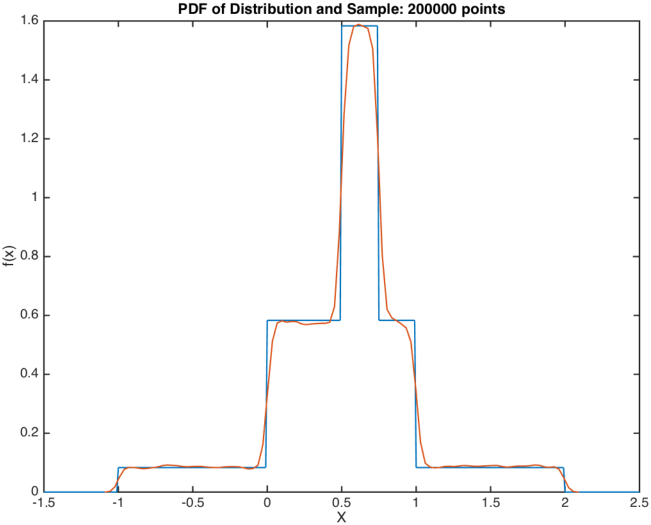
Metropolist-Hastings Random Walk: Example



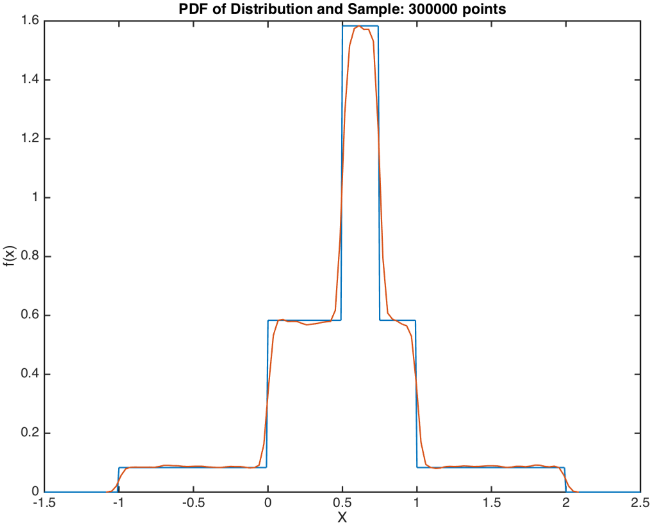
Metropolist-Hastings Random Walk: Example



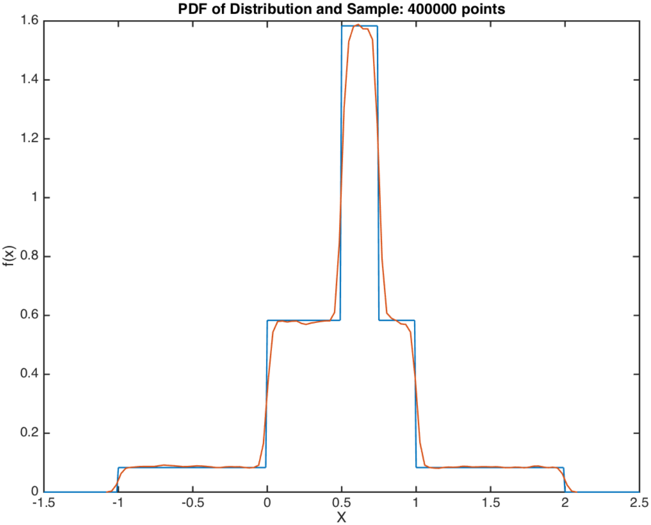
Metropolist-Hastings Random Walk: Example



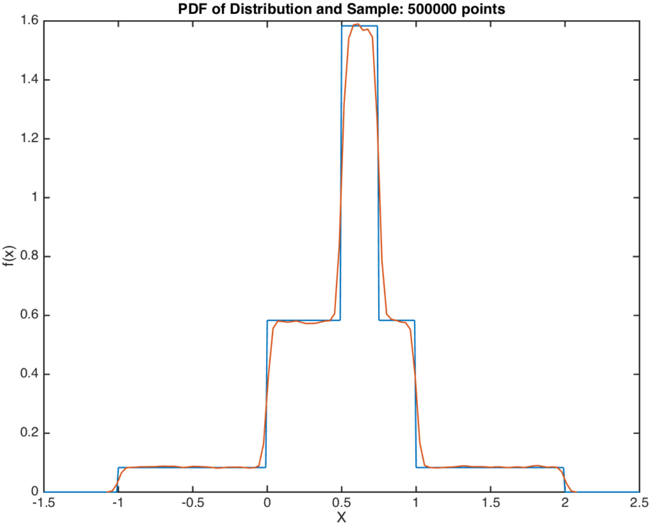
Metropolist-Hastings Random Walk: Example



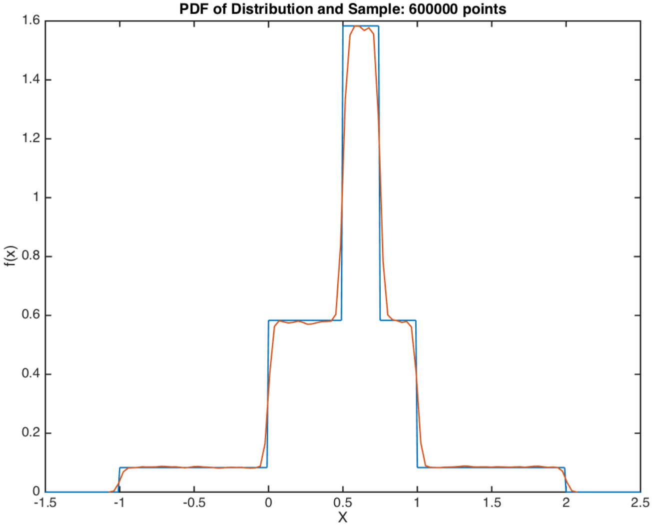
Metropolist-Hastings Random Walk: Example



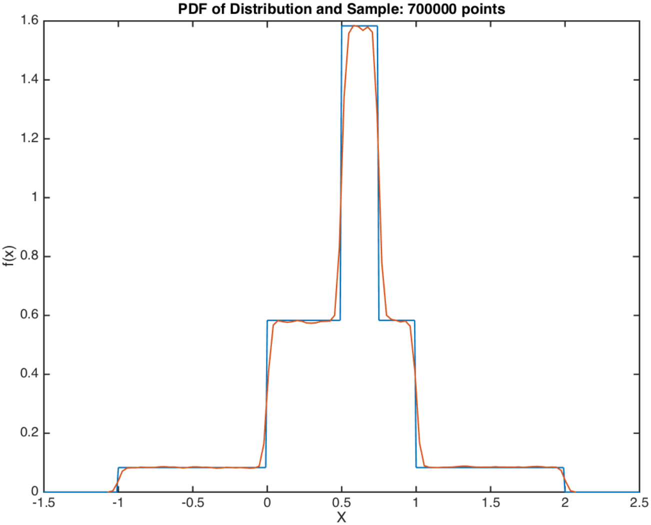
Metropolist-Hastings Random Walk: Example



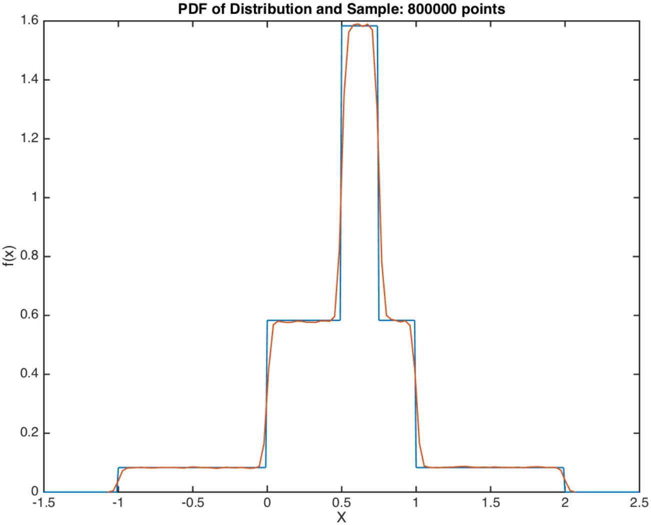
Metropolist-Hastings Random Walk: Example



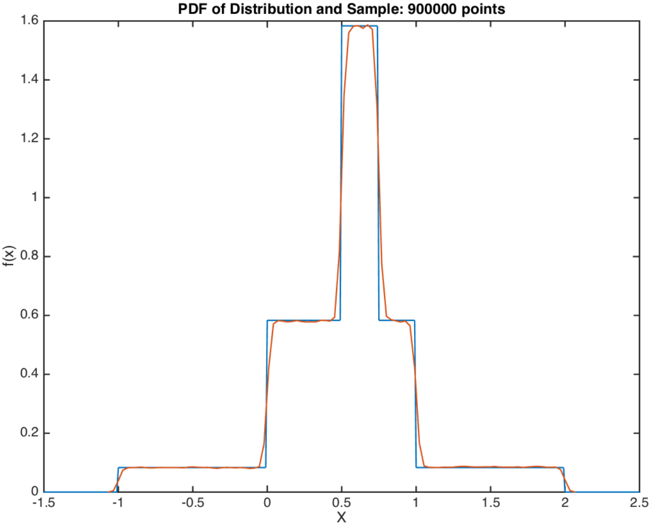
Metropolist-Hastings Random Walk: Example



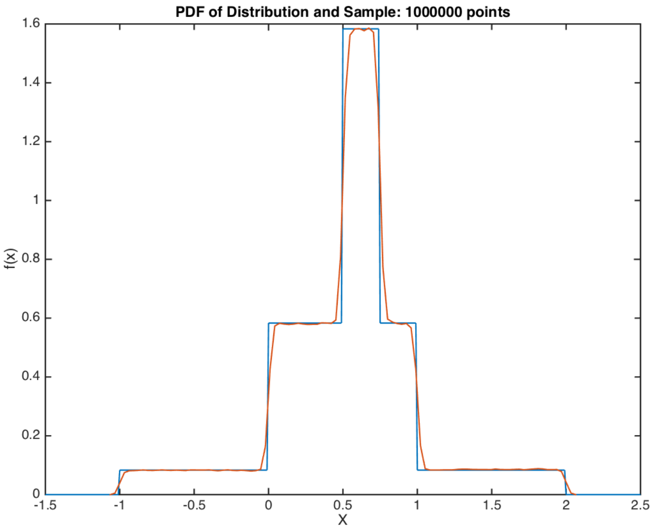
Metropolist-Hastings Random Walk: Example



Metropolist-Hastings Random Walk: Example



Metropolist-Hastings Random Walk: Example



Sampling PDF

- Let's say our distribution is one-dimensional:

$$x \sim 0.5U(0,1) + 0.25U(-1,2) + 0.25U(0.5,0.75)$$

- Choose sampling distribution centered around current point:

$$g(x'|x) \sim \mathcal{N}(x, 0.1)$$

Why learn MH & Numerical Integration?

- Many-dimensional problems
- Bayesian estimation
  - Write down model
  - Write down distribution of parameters $f(\theta)$
  - Simulate many models to get model distribution of data $f(x|\theta)$
  - Update your beliefs: $f(\theta|x) \propto f(\theta)f(x|\theta)$
- Typically need to draw from posterior distribution without an analytical calculation
- Use M-H

Aside: Multiple Hypothesis Testing and Maximum F-Statistics

- Data is tortured. When you see...
  - ..an experiment with multiple test groups or with without very strong theoretical justification, be skeptical!
  - ...a regression that could have been run differently, or with many potential controls, weighting options, and unit-of-observation choices, be skeptical!
- Not all bad: t-statistics might just be heuristics...when I see 0.01 in a regression where I could imagine 10 other setups, I know the real p-value is around 0.1.
- But we might want to take statistics seriously, or data-mine honestly
- We can simulate the distribution of the maximum F-statistic, or the maximum t-statistic.
- See MonteCarlo.do and similar exercises (Note: Stata!)